THE APPLICATION OF RANGE IN THE OVERALL ANALYSIS OF VARIANCE

by

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INTRODUCTION

One of the best known estimators of the variation within a sample is the sample range. Although it has been widely used in industrial quality control, its application to the analysis of experimental data has been limited in favor of the statistically more efficient, but computationally more tedious sample variance. Part of the reason for this is that the analysis of variance was first developed in connection with experiments in which the computational labor of analysis often represented only a small fraction of the labor of experimentation. Hence the maximum amount of information was used in the analysis of the data. Now the analysis of variance is more widely used, but situations frequently arise in which data are cheap and time available for analysis often is limited.

The purposes for which one may wish to use such a short-cut measure are two-fold:

1) In large scale analysis of data one may wish to save computational labor by basing the analysis completely on the short-cut measure.

2) It serves as a quick and independent computational check on a full mean square analysis of variance.

HISTORICAL REVIEW AND RATIONALE

Developments in the use of Range

W. S. Gosset who wrote under the pseudonym of 'Student' is given credit for proposing the use of the ratio of the range divided by an independent estimate s of the population standard deviation. As early as 1932 he referred to this ratio as the 'Studentized' range in a letter written to E. S.

Pearson. In 1944, J. W. Rodgers showed how to utilize the range in estimating all of the variances involved in the analysis of variance and he credited W. J. Jennett for suggesting the procedure. P. B. Patnaik (1950) developed the theory and procedure for the utilization of the range in analysing a completely randomized design. H. O. Hartley (1950) modified the procedure to show how a randomized complete block design might be handled. In 1951, H. A. David presented range analysis methods for analysing the completely randomized design with cell replication and factorial arrangement of treatments, the randomized complete block design with factorial arrangement of treatments, the split-plot design and gave an approximate method of dealing with a completely randomized design with unequal cell frequencies.

The analysis of a Latin square design has not been presented in literature, as it seems to be rather lengthy and hence has few if any advantages over the usual analysis of variance procedure. The range analysis of a Latin square design would also involve a more complex pattern of correlations than those which will arise in the above-mentioned designs.

The purpose of this report is to review the range analysis methods used in the above designs, present them in collected form with an example illustrating the procedure involved in each case and to get an idea of the power which may be attributed to these tests. The data used in the examples in this report have been taken from Snedecor (1961).

Approximation to the Distribution of Mean Range

The following discussion presented by Patnaik (1950) shows how an approximation to the distribution of mean range may be derived. Let $x_1, x_2, \ldots, x_n$ be a random sample of n observations, where $x_{i+1} \geq x_i$, taken from a $N(\mu, \sigma^2)$. Denote the range of this sample by $W_n = x_n - x_1$. Take m indepen-

dent samples, with n observations in each sample, and denote the mean of the m ranges as $\bar{w}_{m,n}$. Denote: σd_n = mean of W_n and $Var(W_n)$ = variance of W_n, where d_n = the population mean of the distribution of range in samples of size n from a N(0,1) population. Then define

$$M = E(\bar{w}_{m,n}/\sigma) = d_n \text{ and} \tag{1}$$

$$V = Var(\bar{w}_{m,n}/\sigma) = (1/m\sigma^2)Var(W_n). \tag{2}$$

As a result of the similarity of the β_1, β_2 points of the distribution of $\bar{w}_{m,n}$ with the β_1, β_2 points of the chi-distribution where β_1, β_2 are measures of skewness and kurtosis respectively, it seems likely that the chi-distribution will give a reasonably accurate representation of the distribution of $\bar{w}_{m,n}$. By equating M and V with the appropriate moments of $c\chi/\sqrt{v}$ where χ has v degrees of freedom one obtains:

$$M = \frac{c}{\sqrt{v}}\sqrt{2}\,\Gamma\left(\frac{v+1}{2}\right)/\Gamma\left(\frac{v}{2}\right), \text{ and} \tag{3}$$

$$V = \frac{c^2}{v}\left[v - 2\left\{\Gamma\left(\frac{v+1}{2}\right)/\Gamma\left(\frac{v}{2}\right)\right\}^2\right] \tag{4}$$

By expanding the Γ-functions by Stirling's formula and solving for v and c one finds the approximations

$$\frac{1}{v} = -2 + 2\sqrt{1 + 2\left[V/M^2 + (1/16)\frac{1}{v^3}\right]}, \text{ and} \tag{5}$$

$$c = M\left[1 + 1/4v + 1/32v^2 - 5/128v^3\right]. \tag{6}$$

Table I in the appendix has been constructed with the aid of equations (5) and (6).

The distribution of $\bar{w}_{m,n}/\sigma$ may be represented by that of $c\chi/\sqrt{v}$; therefore, one may represent the distribution of $\bar{w}_{m,n}/c\sigma$ by that of $\chi/\sqrt{v}$. Since it is known that the ratio of the estimate to the parameter (s/σ) is distributed as $\chi/\sqrt{v}$ it follows that $\bar{w}_{m,n}/c\sigma$ is distributed as s/σ and hence $\bar{w}_{m,n}$ is approximately distributed as cs. The values of M and V may be found

in Pearson's Table A (1932) and by solving the above equations v and then c can be obtained. At present there is no direct method of judging the accuracy of the above approximation.

Distribution of the Range Ratio

Suppose W_n is the range in a sample of n observations from a $N(\mu, \sigma^2)$. Using the s-approximation to the distribution of $\bar{w}_{m,n}/c$ it follows that the range ratio $g = cW_n/\bar{w}_{m,n}$ is distributed approximately as the 'Studentized' range $q = W_n/s$, which is the ratio of a range in a sample of n observations divided by an independent estimate of σ based on v degrees of freedom. The probability integral and percentage points of the 'Studentized' range ratio have been tabled by Pearson & Hartley (1942).

Discussion of Sample Range

The sample range is defined as the difference between the largest and the smallest observation in the sample. It is known that the distribution of the range in normal samples is independent of the population mean, but is dependent on the sample size n and on the population standard deviation, σ. The distribution of range is changed considerably by small departures from normality in the tails of the parental distribution. Also the relative efficiency of the range as an estimator of σ decreases as n increases. Hence in practice if n is large it is preferable to divide the sample into a number of groups and take a weighted mean of the several group ranges. An estimator which is unbiased and possesses minimum variance is defined to be the best unbiased estimator. On this basis Grubbs and Weaver (1947) found that the best unbiased estimate of the population standard deviation, σ, may be obtained by using the mean of group ranges from equal groups of size eight, although

groups of any size between six and ten are nearly as good as eight.

COMPLETELY RANDOMIZED DESIGN

Theoretical Basis

Consider the model

$$X_{ij} = \mu + A_i + \epsilon_{ij} \quad (i = 1,\ldots,m;\ j = 1,\ldots,n) \tag{7}$$

where μ = a constant,

A_i = treatment effects, and

ϵ_{ij} = random error from $N(0,\sigma^2)$.

Patnaik (1950) has shown that the following procedure based on 'Studentized' range may be used in both fixed and random models. It can be shown that W_m, the range of the group means, and $\bar{w}_{m,n}$ the mean of the m group ranges, are statistically independent. Hence the ratio $g = W_m/(\bar{w}_{m,n}/c)$ is approximately distributed as W_m/s where s is independent of W_m. Now W_m is 'Studentized' when it is divided by $s/\sqrt{n}$, the estimate of $\sigma/\sqrt{n}$, which is the standard deviation of the observed group means. Hence $W_m/\left[(\bar{w}_{m,n}/c)/\sqrt{n}\right] = c\sqrt{n}\,W_m/\bar{w}_{m,n}$ is a 'Studentized' range of a sample of size m with degrees of freedom v. Thus $q = W_m c\sqrt{n}/\bar{w}_{m,n}$ is the 'range test' for the hypothesis of no variability between groups. The critical point is the $100\alpha\%$ point of the 'Studentized' range which is found by entering tables of the 'Studentized' range with m groups and v degrees of freedom.

Summary Table for a C.R.D.

Source	D.F.	Sample size	Function calculated
a) Among groups	--	m	$\sqrt{n}\,W_m$
b) Within groups	v	--	$\bar{w}_{m,n}/c$

Illustration of Procedure

Example 1: The following table from (Snedecor, 1961, p. 242) gives the grams of fat absorbed by six batches of doughnuts in each of four fats.

Table 1

Grams of Fat Absorbed by 6 Batches of Doughnuts in Each of 4 Fats (100 grams subtracted from each batch)

	Fats			
	1	2	3	4
	64	78	75	55
	72	91	93	66
	68	97	78	49
	77	82	71	64
	56	85	63	70
	95	77	76	68
Means	72	85	76	62
Ranges	39	20	30	21

Here $m = 4$ fats and $n = 6$ observations per fat.

$W_4 = 85 - 62 = 23$ = range of fat means.

$\bar{w}_{4,6} = (39 + 20 + 30 + 21)/4 = 110/4 = 27.5$ is the average within fat range.

For $m = 4$ and $n = 6$, it can be found from Table I in the appendix that $v = 18.1$ and $c = 2.57$; therefore the following table may be constructed.

Summary Table for Example 1

Source	D.F.	Sample size	Function calculated
a) Among fats	--	4	$\sqrt{6}\,(23)$
b) Within fats	18.1	--	$27.5/2.57$

To test the null hypothesis that all four fats have the same capability for being absorbed by doughnuts against the alternative hypothesis that the fats have different capabilities for being absorbed by doughnuts compute

$$q = a/b = \sqrt{6}\,(23)(2.57)/27.5 \doteq 5.27$$

which is greater than the upper 1% limit of q given in 'Studentized' range tables for $m = 4$ groups and $v = 18.1$ degrees of freedom. The F-test of the analysis of variance gives

$$F = 545.5/100.9 = 5.41$$

which is also significant at the 1% level; therefore, both tests conclude that the fats have different capability for being absorbed by doughnuts.

RANDOMIZED COMPLETE BLOCK DESIGN

Distribution Theory for the Randomized Block Analysis by Range

Considering the model discussed by Hartley (1950) one has

$$X_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \quad (i = 1,\ldots,n \text{ treatments};\ j = 1,\ldots,k \text{ blocks}) \quad (8)$$

where

$$\mu = \text{a constant,}$$

$$\alpha_i = \text{treatment effects,}$$

$$\beta_j = \text{block effects, and}$$

$$\epsilon_{ij} = \text{random error from } N(0,\sigma^2).$$

Form the residuals

$$X_{ij} - \bar{X}_{i.} = \beta_j - \bar{\beta}_{.} + \epsilon_{ij} - \bar{\epsilon}_{i.}$$

Note that for a fixed block, say the jth block, the range W_j of the n values of $X_{ij} - \bar{X}_{i.}$ is equal to the range of the n independent random variables $\epsilon_{ij} - \bar{\epsilon}_{i.}$. In this case $\epsilon_{ij} - \bar{\epsilon}_{i.}$ has zero mean and a variance equal to $\sigma^2 - \sigma^2/k$, where σ^2/k is $\mathrm{cov}(X_{ij}, X_{ij'})$.

The expected value of the range can be determined by

$$E(W_i) = E(\bar{w}_{k,n}) = \sqrt{\sigma^2(1 - 1/k)}\,d_n \qquad (10)$$

where d_n is the population mean of the distribution of range from a $N(0,1)$ population.

The block ranges W_j of $\epsilon_{ij} - \bar{\epsilon}_{i.}$ are not independent, so the distribution results derived for the mean $\bar{w}_{m,n}$ of independent sample ranges by Patnaik (1950) are not appropriate in this case. In this case the expected value of the mean is given by (10) and by using the chi-approximation which Patnaik employed, Hartley (1950) found a similar approximation for the variance.

$$\mathrm{Var}(\bar{w}_{k,n}) = k^{-1}V_n\ (\sigma^2 - \sigma^2/k)\left[1 + (k-1)\rho_w\right] \tag{11}$$

where V_n is the variance of range in samples of size n from a normal population with unit standard deviation and ρ_w is the correlation between any two of the block ranges. If an argument similar to that following equations (1) and (2) is employed for (10) and (11), Table II in the appendix may be constructed.

The 'Studentized' Range Test as a Substitute for the F-test

The test statistic used to test the significance of treatment differences is

$$q = \frac{W_t}{(\bar{w}_{k,n}/c)/\sqrt{k}} = \sqrt{k}\, W_t/(\bar{w}_{k,n}/c)$$

where W_t is the range of treatment means, k is the number of blocks, and $\bar{w}_{k,n}/c$ is an independent estimate of σ. This ratio is referred to tables of the 'Studentized' range, which is the ratio of a range in a sample of n observations divided by an independent estimate of σ based on v degrees of freedom. It has been shown by Hartley (1950) that $\bar{w}_{k,n}/c$ is independent of the ranges of treatment means and block means. The theory behind testing the significance of block differences is similar.

Note that the scale factor c used above was obtained by equating $E(\bar{w}_{k,n})$ and $\mathrm{Var}(\bar{w}_{k,n})$ to the corresponding moments of $c\,\chi/\sqrt{v}$ with χ based on v degrees

of freedom. The difference between this and the previous approximation for c is that $\bar{w}_{k,n}$ is a mean of correlated ranges while $\bar{w}_{m,n}$ is a mean of independent ranges.

Summary Table for R.C.B. Design

	Source	D.F.	Sample size	Function calculated
a)	Treatments	---	n	$\sqrt{k}\, W_t$
b)	Blocks	---	k	$\sqrt{n}\, W_b$
c)	Error	v	---	$s_w = \bar{w}_{k,n}/c$

To test treatment effect: refer a/c to tables of q with sample size n and degrees of freedom v.

To test block effect: refer b/c to tables of q with sample size k and degrees of freedom v.

The small loss in efficiency in using $\bar{w}/c$ in place of s can be estimated by comparing the degrees of freedom of v with the error degrees of freedom of the corresponding full analysis of variance $(n - 1)(k - 1)$. A similar loss of efficiency results when the treatment mean square is replaced by the range of the treatment means.

Illustration of Procedure

Example 2: The analysis of the double classification using ranges may be illustrated on data given by Snedecor (1961, p. 302), on four strains of wheat planted in five randomized blocks.

Table 2

Yields of Four Strains of Wheat in Pounds per Plot

	Strain				
Block	A	B	C	D	Means
1	32.3	33.3	30.8	29.3	31.4
2	34.0	33.0	34.3	26.0	31.8
3	34.3	36.3	35.3	29.8	33.9
4	35.0	36.8	32.3	28.0	33.0
5	36.5	34.5	35.8	28.8	33.9
Means	34.4	34.8	33.7	28.4	

Utilizing the range method one can estimate the error standard deviation s_w. First form differences of individual yields from their respective strain means (strain residuals) and then form the ranges of these residuals for each of the five blocks.

Table 3

Strain Residuals and their Block Ranges

Block	A	B	C	D	Range
1	-2.1	-1.5	-2.9	0.9	3.8
2	-0.4	-1.8	0.6	-2.4	3.0
3	-0.1	1.5	1.6	1.4	1.7
4	0.6	2.0	-1.4	-0.4	3.4
5	2.1	-0.3	2.1	0.4	2.4
Total					14.3

The estimate of the error standard deviation is obtained from the mean range by the simple equation

$$s_w = \bar{w}_{k,n}/c$$

where $\bar{w}_{k,n}$ is the mean range of the blocks and c is a scale factor obtained from Table II in the appendix, with k = number of blocks = 5 and n = number of treatments = 4. This gives

$$s_w = (14.3/5)/1.88 = 1.52.$$

The table also shows the equivalent degrees of freedom or v = 10.9 on which the estimate s_w is based. Comparing this with the error degrees of freedom (12) from the analysis of variance indicates a loss of 1.1 degrees of freedom with this approximate method.

In order to test the significance of treatment differences replace the computation of the treatment mean square by the range of the treatment means and utilize the equation

$$q = \sqrt{k}\, W_t/s_w$$

where k is the number of blocks and W_t is the range of the treatment means.

Thus consider:

$$q = \sqrt{5}\,(34.8 - 28.4)/1.52 = 9.4$$

Looking in tables of the 'Studentized' range for n = number of treatments = 4, and v = 10.9 one may find $q_{.01} = 5.5$ which is less than the observed value of 9.4 indicating highly significant treatment differences as was found with the analysis of variance test where it was shown that

$$F = 44.82/2.19 = 20 \text{ where}$$

$$F_{.01;\,4,3} = 5.95.$$

If a test for block differences is desired one may utilize the equation

$$q = \sqrt{n}\,W_b/s_w = \sqrt{4}\,(33.9 - 31.4)/1.52 \doteq 3.7$$

where n is the number of treatments and W_b is the range of block means. Entering the 'Studentized' range table with n = 5 and v = 10.9 one finds that $q_{.05} = 4.6$ so there are no significant block differences. Similar results are obtained from the analysis of variance in Snedecor where

$$F = 5.36/2.19 = 2.4 \text{ and}$$

$$F_{.05;\,4,12} = 3.26.$$

Summary Table for Example 2

Source	D.F.	Sample size	Function calculated
a) Treatments	--	4	$\sqrt{5}\,(6.4)$
b) Blocks	--	5	$\sqrt{4}\,(2.5)$
c) Error	10.9	--	(14.3/5)/1.88

COMPLETELY RANDOMIZED DESIGN WITH CELL REPLICATION AND FACTORIAL ARRANGEMENT OF TREATMENTS

When utilizing the range method it is necessary to consider both the fixed and random models in this design just as would be done in the usual analysis of variance procedure.

Fixed Model

Consider the model discussed by David (1951) where

$$X_{ijt} = \mu + A_i + B_j + (AB)_{ij} + \epsilon_{ijt} \quad (i = 1,\ldots,a;\ j = 1,\ldots,b;\ t = 1,\ldots,n) \qquad (12)$$

and μ = a constant,

A_i = fixed effects of treatment A,

B_j = fixed effects of treatment B,

$(AB)_{ij}$ = fixed interaction effects, and

ϵ_{ijt} = random error from $N(0,\sigma^2)$, and note that $\sum_{i=1}^{a} A_i = \sum_{j=1}^{b} B_j = \sum_{i=1}^{a} (AB)_{ij} = \sum_{j=1}^{b} (AB)_{ij} = 0$. The ϵ_{ijt} are independent normal random variables distributed as $N(0,\sigma^2)$. The statistic $w_n(i,j)$ can be computed as the range of the individual observations X_{ijt} and its distribution may be regarded as that of the range of the random variables ϵ_{ijt} as may be seen from

$$w_n(i,j) \equiv \text{range } (X_{ijt}) = \text{range } (\epsilon_{ijt}) \text{ for fixed } i,j \text{ and } t = 1,\ldots,n. \qquad (13)$$

Recall that from Patnaik's work (1950) an approximation to the distribution of the mean $\bar{w}$ of a set of independent ranges has been obtained by equating the expectation M and variance V of $\bar{w}/\sigma$ to those of $c\,\chi/\sqrt{\nu}$. It is permissible to enter an extension of Patnaik's table with sample size n, and number of samples m = ab, to give $s_w = \bar{w}/c$, the estimator of σ.

Now from (12) consider

$$\bar{X}_{ij.} = \mu + A_i + B_j + (AB)_{ij} + \bar{\epsilon}_{ij.} \qquad (14)$$

and corresponding to (13) note that

$$w_a'(j) \equiv \text{range } (\bar{X}_{ij.} - \bar{X}_{i..}) = \text{range } ((AB)_{ij} + \bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..}) \text{ for fixed } j.$$

The b different ranges w_a' are not independent and for the null hypothesis

$$Ho:\ (AB)_{ij} = 0 \ (i = 1,\ldots,a;\ j = 1,\ldots b),$$

Hartley (1950) obtained the mean and variance of $\bar{w}'$ and used these to fit a similar chi-approximation to the mean of the correlated ranges. It can be seen that $s_w' = (\bar{w}'/c')/\sqrt{n} = \bar{w}'\sqrt{n}/c'$ is an estimator of σ regardless of whether A_i and / or B_j are present if the null hypothesis concerning no interaction is true. From the alternative hypothesis of inequality it can be seen from above that s_w' will depend on the interaction terms $(AB)_{ij}$ and the ϵ_{ijt} only. It has been shown by David (1951) that s_w and s_w' are independent, so under the null hypothesis the ratio $s_w'^2/s_w^2$ is distributed approximately as F with degrees of freedom v', v. Since $\bar{X}_{i..} = \mu + A_i + \epsilon_{i..}$, the usual 'Studentized' range criterion may be used to test the main effects; thus

$$q_A = \text{range } (\bar{X}_{i..})/(s_w/\sqrt{bn}) = \sqrt{bn} \text{ range } (\bar{X}_{i..})/s_w$$

and

$$q_B = \text{range } (\bar{X}_{.j.})/(s_w/\sqrt{an}) = \sqrt{an} \text{ range } (\bar{X}_{.j.})/s_w.$$

Random Model

Consider the model

$$X_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijt} \quad (i = 1,\ldots,a;\ j = 1,\ldots,b;\ t = 1,\ldots,n) \quad (15)$$

where μ = a constant,

α_i = random effects of treatment A,

β_j = random effects of treatment B,

$(\alpha\beta)_{ij}$ = random interaction effects, and

ϵ_{ijt} = random error from $N(0,\sigma^2)$.

In the model α_i, β_j, $(\alpha\beta)_{ij}$, and ϵ_{ijt} are all independent normal random variables with zero means and respective variances σ_A^2, σ_B^2, σ_{AB}^2, and σ^2. The development of the theory is much the same as above, except that main effects must now be tested against s_w', for instance

$$q_A = \text{range } (\bar{X}_{i..})/(s_w'/\sqrt{bn}) = \sqrt{bn} \text{ range } (\bar{X}_{i..})/s_w'.$$

Summary Table for Completely Randomized Design with Cell Replication and Factorial Arrangement of Treatments

Source	D.F.	Sample size	Function calculated
a) Treatment A	--	a	$\sqrt{bn}$ range $(\bar{X}_{i..})$
b) Treatment B	--	b	$\sqrt{an}$ range $(\bar{X}_{.j.})$
c) Interaction	v'	--	$s_w' = \bar{w}'\sqrt{n}/c'$
d) Residual	v	--	$s_w = \bar{w}/c$

Tests:

For treatment A:

Random model: refer a/c to tables of q with sample size a and degrees of freedom = v'.

Fixed model: refer a/d to tables of q with sample size a and degrees of freedom = v.

For treatment B:

Random model: refer b/c to tables of q with sample size b and degrees of freedom v'.

Fixed model: refer b/d to tables of q with sample size b and degrees of freedom v.

For interaction: refer, on either model, c^2/d^2 to tables of F with degrees of freedom v', v.

Illustration of Procedure

Example 3: To illustrate the procedure consider the following data taken from (Snedecor, 1961, p. 340) which may be described by the fixed model since treatment effects are assumed to be fixed.

Table 4

Treatment A	Treatment B b_1	b_2	b_3
a_1	26	30	49
	38	39	47
a_2	43	29	30
	40	28	35
a_3	28	29	36
	33	12	37
a_4	15	16	17
	17	17	12

Table 5

Totals of Two Replicates

Treatment A	Treatment B b_1	b_2	b_3	Total
a_1	64	69	96	229
a_2	83	57	65	205
a_3	61	41	73	175
a_4	32	33	29	94
Total	240	200	263	

Table 6

Ranges of Two Replicates

Treatment A	Treatment B b_1	b_2	b_3
a_1	12	9	2
a_2	3	1	5
a_3	5	17	1
a_4	2	1	5

To compute the 'treatment B residuals', it is necessary to subtract from each observation in Table 5 one-fourth of the column total from which the observation originated. Thus the following table is derived.

Table 7

Treatment B Residuals (total)

Treatment A	b_1	b_2	b_3	Range
a_1	4	19	30	26
a_2	23	7	- 1	24
a_3	1	- 9	7	16
a_4	-28	-17	-37	20
Total				86

To estimate the residual error one must first find

$$\bar{w} = (12 + \ldots + 5)/12 = 63/12 = 5.25$$

from which it follows that

$$s_w = \bar{w}/c = 5.25/1.16 \doteq 4.53$$

where c is found by entering Table I in the appendix, with $n = 2$ replications and $m = ab = 12$. This value has the equivalent degrees of freedom $v = 9.0 + 2(.88) = 10.8$. In comparison note that Snedecor found $s = \sqrt{25} = 5$ with 12 degrees of freedom.

To estimate interaction utilize Table (7) and compute

$$\bar{w}' = 86/8 = 10.75.$$

Then one has

$$s_w' = \bar{w}'\sqrt{n}/c' = 10.75\sqrt{2}/1.54 \doteq 9.87$$

where c' is found by entering Table II in the appendix, with $n = 3$ treatment B factors and $k = 4$ ranges. The equivalent degrees of freedom are $v' = 5.4$. In comparison Snedecor found interaction error $= \sqrt{81} = 9$ with 6 degrees of freedom.

Refer

$$F = s_w'^2/s_w^2 = (9.87)^2/(4.53)^2 \doteq 4.75$$

to tables of the F-ratio with degrees of freedom $v' = 5.4$, $v = 10.8$ and interaction is found to be significant at the .05 level. To test for treatment A and treatment B effects form 'Studentized' ranges q_A and q_B. Thus:

$$q_A = \sqrt{an}\ \text{range}(\bar{X}_{i..})/s_w = \frac{\sqrt{8}\ (229 - 94)/8}{4.53} \doteq 10.5 \text{ and}$$

$$q_B = \sqrt{bn}\ \text{range}\ (\bar{X}_{.j.})/s_w = \frac{\sqrt{6}\ (263 - 200)/6}{4.53} \doteq 5.7.$$

It can be seen that q_A is significant at the .01 level and q_B is significant at the .05 level. These three tests correspond to those made in the usual analysis of variance and give similar results. Note that in contrast to the usual procedure, the range method is little affected by an increase in the number of replications.

Summary Table for Above Example

Source	D.F.	Sample size	Numerical value of function
a) Treatment A	--	4	$\sqrt{8}\ (135)/8$
b) Treatment B	--	3	$\sqrt{6}\ (63)/6$
c) Interaction	5.4	--	$(86/8)\sqrt{2}/1.54$
d) Residual	10.8	--	$5.25/1.16$

RANDOMIZED COMPLETE BLOCK DESIGN WITH FACTORIAL ARRANGEMENT OF TREATMENTS

Theoretical Basis

Consider the theoretical model in the fixed form discussed by David (1951). Thus

$$X_{ijt} = \mu + A_i + B_j + (AB)_{ij} + C_t + \epsilon_{ijt} \quad (i = 1,\ldots,l;\ j = 1,\ldots,m;\ t = 1,\ldots,n) \tag{16}$$

where μ = a constant,

A_i = fixed effects of treatment A,

B_j = fixed effects of treatment B,

$(AB)_{ij}$ = fixed interaction effects,

C_t = fixed block effects, and

ϵ_{ijt} = random error from $N(0,\sigma^2)$.

The treatment combinations denoted by the suffices (i,j) may be broken up into two treatments and their interaction and t is the block index.

It follows from considering (16) that

$$\bar{X}_{ij.} = \mu + A_i + B_j + (AB)_{ij} + \bar{\epsilon}_{ij.} \tag{17}$$

where the residual error may be estimated in the same way as the interaction (blocks x treatment combinations) in an lm x n ordinary randomized complete block example and the interaction term is estimated as in the previous example.

Summary Table for Randomized Complete Block Design
With Factorial Arrangement of Treatments

Source	D.F.	Sample size	Function calculated
a) Treatment A	--	l	$\sqrt{mn}$ range $(\bar{X}_{i..})$
b) Treatment B	--	m	$\sqrt{ln}$ range $(\bar{X}_{.j.})$
c) Blocks C	--	n	$\sqrt{lm}$ range $(\bar{X}_{..t})$
d) Interaction AxB	v_2'	--	$s_{w_2}' = \bar{w}_2'\sqrt{n}/c_2$
e) Residual	v_1'	--	$s_{w_1}' = \bar{w}_1/c_1'$

Tests:

For treatment A

With treatment B random: refer a/d to tables of q with sample size l and degrees of freedom v_2'.

With treatment B fixed: refer a/e to tables of q with sample size l and degrees of freedom v_1'.

For blocks C: refer c/e to tables of q with sample size n and degrees of freedom v_1' for either model.

For interaction: refer d^2/e^2 to tables of F with degrees of freedom v_2', v_1'.

Illustration of Procedure

Thus with the above changes in mind the procedure may be illustrated by looking at the following example from (Snedecor, 1961, p. 351). The fixed model is appropriate in this case as treatment effects and interaction effects are fixed.

Example 4: Table 8

Yield of Cowpea Hay in Pounds From 3 Varieties

Variety	Spacing (in.)	Block 1	2	3	4
	4	56	45	43	46
I	8	60	50	45	48
	12	66	57	50	50
	4	65	61	60	63
II	8	60	58	56	60
	12	53	53	48	55
	4	60	61	50	53
III	8	62	68	67	60
	12	73	77	77	65
Block total		555	530	496	500

From Table 8 form the block means $\bar{X}_{..1} = 555/9 \doteq 62$, $\bar{X}_{..2} = 530/9 \doteq 59$, $\bar{X}_{..3} = 496/9 \doteq 55$, $\bar{X}_{..4} = 500/9 \doteq 56$. Form the block residual table from $X_{ijt} - \bar{X}_{..t}$.

Table 9

Block Residuals and their Treatment Ranges

Variety	Spacing (In.)	Block 1	2	3	4	Treatment ranges
	4	- 6	-14	-12	-10	8
I	8	- 2	- 9	-10	- 8	8
	12	4	- 2	- 5	- 6	10
	4	3	2	5	7	5
II	8	- 2	- 1	1	4	6
	12	- 9	- 6	- 7	- 1	8
	4	- 2	2	- 5	- 3	7
III	8	0	9	12	4	12
	12	11	18	22	9	13
Treatment range total						77

Entering Table II in the appendix, with k = 9 ranges and n = 4 blocks

it may be found that $c_1' = 1.96$ and $v_1' = 21.7$. Thus

$$s_{w_1}' = \bar{w}_1'/c_1' = (77/12)(1/1.96) \doteq 3.4$$

which corresponds to Snedecor's $s = \sqrt{17.67} \doteq 4.2$ with 24 degrees of freedom.

To calculate s_{w_2}' it is necessary to compute the following two tables.

Table 10

Totals of 4 Replicates

	Spacing (in.)			
Varieties	4	8	12	Variety totals
I	190	203	223	616
II	249	234	209	692
III	224	257	292	773
Spacing totals	663	694	724	

The spacing (residuals) in the following table were computed by subtracting from each observation in Table (10) one-third of the column total from which the observation originated.

Table 11

Spacing (Residuals)

	Spacing (in.)			
Varieties	1	2	3	Range
I	-31	-28	-18	13
II	28	3	-32	60
III	3	26	51	48
Total				121

Now for $n = 3$ spacings and $k = 3$ ranges one may enter Table II in the appendix, and find $c_2' = 1.48$ and $v_2' = 3.7$. Thus

$$s_{w_2}' = \bar{w}_2'\sqrt{n}/c_2' = 121\sqrt{4}/(12)(1.48) \doteq 13.6.$$

The ratio

$$F = s_{w_2}'^2/s_{w_1}'^2 = (13.6)^2/(3.4)^2 = 16$$

is highly significant for $v_2' = 3.7$ and $v_1' = 21.7$. Snedecor also found this ratio to be highly significant.

To finish the analysis compute

$$q_A = \sqrt{mn}\ \text{range}\ (\bar{X}_{i..})/s_{w_1}' = \frac{\sqrt{12}\ (773 - 616)/12}{3.4} \doteq 45.4/3.4 \doteq 13.1,$$

$$q_B = \sqrt{ln}\ \text{range}\ (\bar{X}_{.j.})/s_{w_1}' = \frac{\sqrt{12}\ (724 - 663)/12}{3.4} \doteq 17.6/3.4 \doteq 5.2, \text{ and}$$

$$q_C = \sqrt{lm}\ \text{range}\ (\bar{X}_{..t})/s_{w_1}' = \frac{\sqrt{9}\ (555 - 496)/9}{3.4} \doteq 19.7/3.4 \doteq 5.8.$$

The first and the third test statistic are significant at the .01 level and the second test statistic is significant at the .05 level. Snedecor's analysis gives the same results.

Summary Table for Above Example

Source	D.F.	Sample size	Function calculated
a) Varities	--	3	$\sqrt{12}\ (773 - 616)/12$
b) Spacings (in.)	--	3	$\sqrt{12}\ (724 - 663)/12$
c) Blocks C	--	4	$\sqrt{9}\ (555 - 496)/9$
d) Interaction	3.7	--	$s_{w_2}' = (121\sqrt{4}/12)/1.76$
e) Residual	21.7	--	$s_{w_1}' = (77/12)/1.96$

SPLIT-PLOT DESIGN

Theoretical Basis

Consider the model discussed by David (1951).

$$X_{ijt} = \mu + A_i + B_j + \delta_{ij} + C_t + (AC)_{it} + \epsilon_{ijt}\ (i = 1,\ldots,l;\ j = 1,\ldots,m;\ t = 1,\ldots,n) \qquad (18)$$

where μ = a constant,

A_i = fixed effects of treatment A,

B_j = fixed block effects,

δ_{ij} = random main plot error

C_t = fixed subtreatment effects,

$(AC)_{it}$ = fixed interaction effects, and

ϵ_{ijt} = random sub-plot error component.

The only change in this analysis is the way in which the residual, the sub-plot error, is estimated. To estimate this error, σ, note that

$$X_{ijt} - \bar{X}_{ij.} = C_t + (AC)_{it} + \epsilon_{ijt} - \bar{\epsilon}_{ij.},$$

and thus

$w_m(i,t) \equiv$ range $(X_{ijt} - \bar{X}_{ij.})$ = range $(\epsilon_{ijt} - \bar{\epsilon}_{ij.})$, for fixed i, t and $j = 1,\ldots,m$.

Therefore ln correlated ranges $w_m(i,t)$ are obtained.

Patnaik's (1950) approximation to the distribution of the mean of independent ranges by fitting a chi-distribution adjusted to have the correct mean M and variance V has been discussed previously. By making a similar approximation for the mean of correlated ranges, c and v may be determined from his equations

$$V/M^2 = 1/2v + 1/8v^2 - 1/16v^3 + \ldots \tag{19}$$

and

$$c = M(1 + 1/4v + 1/32v^2 - 5/128v^3 + \ldots) \tag{20}$$

provided that appropriate expressions for M and V may be derived for the desired design.

In the split-plot design the correlation is zero between $\epsilon_{ijt} - \bar{\epsilon}_{ij.}$ and $\epsilon_{i'jt'} - \bar{\epsilon}_{i'j.}$ if $i \neq i'$ and is $-1/(n - 1)$ if $i = i'$, so the ln ranges are arranged into l independent groups of size n. Within each group any two ranges are correlated with correlation coefficient ρ_w, which is a function of $\rho = -1/(n - 1)$ and the sample size m. Hence David found that

$$M = E(\bar{w}) = d_m \sigma \sqrt{(1 - 1/n)}, \tag{21}$$

$$V = Var(\bar{w}) = Var(\sum_i \sum_t w_m(i,t)/ln, \tag{22}$$

$$= [1 + (n - 1)\rho_w] V_m \sigma^2 (1 - 1/n)/ln,$$

and

$$V/M^2 = (1/ln)(V_m/d_m^2) [1 + (n - 1)\rho_w(n,m)] , \tag{23}$$

where d_m is the expectation and V_m the variance of the range of m independent unit normal random variables. From (19), (20), (21), and (22) David constructed the Table which he utilized in analysing split-plot experiments. This table appears as Table III in the appendix.

Summary Table for the Split-plot Design

Source	D.F.	Sample size	Function calculated
a) Main treatment A	--	1	$\sqrt{mn}$ range $(\bar{X}_{i..})$
b) Blocks B	--	m	$\sqrt{ln}$ range $(\bar{X}_{.j.})$
c) Main plot error	v' (Table II)	--	$s_w' = \bar{w}'\sqrt{n}/c'$
d) Subtreatment C	--	n	$\sqrt{lm}$ range $(\bar{X}_{..t})$
e) Interaction AxC	v_1' (Table II)	--	$s_1' = \bar{w}_1'\sqrt{m}/c_1'$
f) Sub-plot error	v (Table III)	--	$s_w = \bar{w}/c$

Tests:

For treatment A: refer a/c to tables of q with sample size 1 and degrees of freedom v'.

For subtreatment C: refer d/f to tables of q with sample size n and degrees of freedom v.

For main plot error: refer c^2/d^2 to tables of F with degrees of freedom v', v.

Illustration of Procedure

The following example is taken from Snedecor (1961, p. 367).

Example 5:

Table 12

Yields of Variety 1

Subtreatment	Block 1	2	3	4	5	6	Mean
1	2.17	1.88	1.62	2.34	1.58	1.66	1.88
2	1.58	1.26	1.22	1.59	1.25	0.94	1.31
3	2.29	1.60	1.67	1.91	1.39	1.12	1.66
4	2.23	2.01	1.82	2.10	1.66	1.10	1.82

Note that Table (12) has listed only one of the three varieties being considered.

Table 13

Variety-subtreatment Means

Subtreatment	Variety (main treatment) 1	2	3	Mean
1	1.88	1.76	1.70	1.78
2	1.31	1.30	1.41	1.34
3	1.66	1.58	1.48	1.57
4	1.82	1.64	1.61	1.69
Mean	1.67	1.57	1.55	

Table 14

Subtreatment Residuals of Table 12 and Their Block Ranges

Subtreatment	Block 1	2	3	4	5	6
1	0.29	0.00	-0.26	0.46	-0.30	-0.22
2	0.27	-0.05	-0.09	0.28	-0.06	-0.37
3	0.63	-0.06	0.01	0.25	-0.27	-0.54
4	0.41	0.19	0.00	0.28	-0.16	-0.72
Range	0.36	0.25	0.27	0.21	0.24	0.50

Table (14) was computed by subtracting from each observation in Table (12) its respective subtreatment mean. Table (15) was computed by subtracting the variety mean from each observation in Table (13).

Table 15

Table for the Estimation of the Variety-subtreatment Interaction

Subtreatment	Variety 1	2	3	Range
1	0.21	0.19	0.15	0.06
2	-0.36	-0.27	-0.14	0.22
3	-0.01	0.01	-0.07	0.08
4	0.15	0.07	0.06	0.09
Total				0.45

This example has $l = 3$ varieties, $m = 6$ blocks, $n = 4$ subtreatments and $\bar{w}$ is the mean of eighteen ranges the first six of which are listed in Table (14). Entering Table III in the appendix, with $n = 4$ and $m = 6$, we find $c = 1.90$ and $v = 3(13.4) = 40.2$ where 3 is the number of varieties.

Thus:

$$s_w = \bar{w}/c = .310/1.90 = .163$$

In the analysis of variance Snedecor found $s = .167$ with 45 degrees of freedom. To estimate the variety-subtreatment interaction utilize Table(15) and find $\bar{w}_1' = .45/4$. Entering Table II in the appendix, with $n = 3$ observations and $m = k = 4$ ranges, note that $c_1' = 1.54$ and $v_1' = 5.4$. Hence

$$s_1' = \bar{w}_1'\sqrt{m}/c_1' = (.45/4)\sqrt{6}/1.54 = .179, \text{ and}$$

$$F_{5.4,40.2} = s_1'^2/s_w^2 = (.179)^2/(.163)^2 = 1.21$$

corresponding to Snedecor's $F_{6,45} = 1.25$.

To estimate the main plot error we must construct the following two tables.

Table 16

Totals of 6 Replications

	Subtreatments				
Varieties	1	2	3	4	Variety totals
1	11.25	7.84	9.86	10.92	39.99
2	10.59	7.81	9.46	9.86	37.72
3	10.22	8.48	8.90	9.66	37.26
Subtreatment totals	32.06	24.13	28.34	30.44	114.97

To compute the subtreatment residuals it is necessary to subtract from each observation in Table(16) one-third of the column total from which the observation originated. Thus the following table may be constructed.

Table 17

Subtreatment Residuals

	Subtreatments				
Varieties	1	2	3	4	Variety range
1	.56	-.20	.54	.78	.98
2	-.10	-.23	.02	.28	.51
3	-.47	.44	-.54	.48	1.02
Total					2.51

To estimate the main plot error utilize Table (17) and find $\bar{w}_1' = 2.51/3$. Entering Table II in the appendix, with $n = 6$ replications and $k = 3$ ranges,

note that $c' = 2.12$ and $v' = 9.3$. Hence

$$s_w' = \bar{w}'\sqrt{n}/c' = \frac{(2.51/3)\sqrt{4}}{2.12} \doteq .79.$$

Compute the test statistics

$$q_A = \sqrt{mn}\ \text{range}\ (\bar{X}_{i..})/s_w' = \sqrt{24}\ (1.67 - 1.55)/.79 \doteq .7 \text{ and}$$

$$q_C = \sqrt{lm}\ \text{range}\ (\bar{X}_{..t})/s_w = \sqrt{18}\ (1.78 - 1.34)/.163 \doteq 11.51.$$

Variety effects are thus found to be nonsignificant while subtreatment effects are significant at the .01 level. The results above are verified by Snedecor's analysis.

Summary Table for the Above Example

Source	D.F.	Sample size	Function calculated
a) Main treatment A	--	3	$\sqrt{24}\ (1.67 - 1.55)$
b) Blocks B	--	6	$\sqrt{12}\ (.61 - .09)$
c) Main plot error	9.3	--	$s_{w_1}' = (2.51/3)\sqrt{4}/2.12$
d) Subtreatment C	--	4	$\sqrt{18}\ (1.78 - 1.34)/.163$
e) Interaction AxC	5.4	--	$s_1' = (.45/4)\sqrt{6}/1.54$
f) Sub-plot error	40.2	--	$s_w = (.558/18)/1.90$

COMPLETELY RANDOMIZED DESIGN WITH UNEQUAL CELL FREQUENCIES

The accuracy of the approximations used and the labor saved (in comparison with the customary mean square procedure) will in general increase markedly with the total number of observations, which should be greater than 20. Three methods of dealing with the present case are:

a) the standard technique based on the ratio of two sums of squares,

b) a method in which the between-group estimate of (a) is divided by a new within-group estimate based on a 'weighted mean range' and

c) the 'unweighted mean range' method in which both the between and within group estimates are based on range.

The last method involves the q-ratio employed by Patnaik for the case of nearly equal cell frequencies and is valid only in the case of nearly equal

cell frequencies.

The following example is taken from (Snedecor, 1961, p. 269).

Example 6: Table 18

Birth Weight (pounds) of Poland China Pigs in Eight Litters

Litter	Birth Weights	Litter size	Mean	Range
1	2.0, 2.8, 3.3, 3.2, 4.4 3.6, 1.9, 3.3, 2.8, 1.1	10	2.84	3.3
2	3.5, 2.8, 3.2, 3.5, 2.3 2.4, 2.0, 1.6	8	2.66	2.1
3	3.3, 3.6, 2.6, 3.1, 3.2 3.3, 2.9, 3.4, 3.2, 3.2	10	3.18	.8
4	3.2, 3.3, 3.2, 2.9, 3.3, 2.5, 2.6, 2.8	8	2.98	.8
5	2.6, 2.6, 2.9, 2.0, 2.0 2.1	6	2.37	.9
6	3.1, 2.9, 3.1, 2.5	4	2.90	.6
7	2.6, 2.2, 2.2, 2.5, 1.2	6	1.98	1.4
8	2.5, 2.4, 3.0, 1.5	4	2.35	1.5
Total		56		11.4

Method a) Snedecor found litter mean square = 1.07 and residual mean square, $s^2 = .36$. Hence $s = .6$ with 48 degrees of freedom.

Method b) The residual mean square estimate is given by

$$s_w = \Sigma(W_{n_i} d_{n_i}/V_{n_i})/[\Sigma(d_{n_i}{}^2/V_{n_i}) + .5]$$

with $v = \frac{1}{2}\Sigma(d_{n_i}{}^2/V_{n_i})$ where d_{n_i} and V_{n_i} are the expectation and variance of the range in samples of n_i independent unit normal random variables. The quantities d_{n_i}/V_{n_i} are given by David for n_i = 2-20. This is also reproduced as Table IV in the appendix. Thus

$$s_w = (3.3 \times 4.84 + \ldots + 1.5 \times 2.66)/(14.90 + \ldots + 5.48 + .5)$$

$\doteq .55$ with 41.4 degrees of freedom.

The numerator in the F-ratio is calculated as in (a).

Method c) The estimate of residual error is $\bar{w}/c = (11.4/8)/2.72 \doteq .52$ with $v = 42.4$ where c and v are found by interpolation from Table I in the appendix with $n = \Sigma n_i/8 = 7$ and $k = 8$.

Comparison of Three Methods of Analysing a Simple Classification
With Unequal Cell Frequencies

Method	Value of Test Ratio	Upper 1% point
a) Standard	$F_{7,48} = 1.07/.36 \doteq 2.97$	3.04
b) Weighted mean range	$F_{7,41.4} = 1.07/(.55)^2 \doteq 3.54$	3.11
c) Unweighted mean range	$q_{8,42.2} = 1.20\sqrt{7}/.52 \doteq 6.11$	5.37

MULTIPLE COMPARISONS

Two procedures for mean separation to follow the regular analysis of variance are the multiple t or LSD approach and the multiple range based on the 'Studentized' range. Analogous procedures which one might consider to follow the range analysis of variance are the multiple G-test and a stepwise reduction of the q-test.

Multiple G-test

The G-test simply involves the use of the mean range in place of the standard error in a t-test. When testing the null hypothesis that $\mu_1 = \mu_2 = 0$ the t-test between two means may be represented by

$$t = (\bar{X}_1 - \bar{X}_2)/\sqrt{2s^2/n}.$$

Thus

$$LSD_\alpha = t_\alpha \sqrt{2s^2/n}$$

may be used as a criterion for determining if two means differ significantly. If the same null hypothesis is to be tested following a range analysis it may be seen from

$$G = (\bar{X}_1 - \bar{X}_2)/\bar{w}$$

that

$$\text{'range' } LSD_\alpha = G_\alpha \bar{w}$$

where $\bar{w}$ is the mean of the treatment ranges.

Consider the example illustrated for the completely randomized design. For the doughnuts $\bar{w} = 27.5$. With 6 observations per group and $\alpha = .05$, G is taken from tables of $G = (\bar{X}_1 - \bar{X}_2)/\bar{w}$ to be .499. Thus

$$G_{.05}\, \bar{w} = (.499)(27.5) \doteq 13.72$$

from which differences between individual means may be compared. The following table of mean differences may be computed from the data.

Table 19

Fat	$\bar{X}$	$\bar{X} - 162$	$\bar{X} - 172$	$\bar{X} - 176$
2	185	23	13	9
3	176	14	4	
1	172	10		
4	162			

From the above table and the 'range' LSD computed, it may be concluded that fat 2 is absorbed in greater amounts than fat 4 and fat 3 is absorbed in greater amounts than fat 4, but that there are no detected differences among the remaining four comparisons. It might be noted that in addition to the conclusion that $\mu_2 \neq \mu_4$ and $\mu_3 \neq \mu_4$ the regular LSD also finds $\mu_1 \neq \mu_2$ since the regular LSD $\doteq$ 11.06. The Student-Newman-Keuls formula detects only $\mu_2 \neq \mu_4$.

Stepwise Reduction of the q-test

In this procedure one may rank the differences between two treatment means from highest to lowest and apply the q-test to successive differences until a difference too small to indicate significance is found.

Considering again the example illustrated for the completely randomized design one may set up the following six differences:

$$W_6 = \bar{X}_2 - \bar{X}_4 = 23, \qquad W_3 = \bar{X}_1 - \bar{X}_4 = 10,$$

$$W_5 = \bar{X}_3 - \bar{X}_4 = 14, \qquad W_2 = \bar{X}_2 - \bar{X}_3 = 9,$$

$$W_4 = \bar{X}_2 - \bar{X}_1 = 13, \qquad W_1 = \bar{X}_3 - \bar{X}_1 = 4,$$

and let W_i denote the i th largest difference. Since

$$q = W_i\sqrt{n}/(\bar{w}_{m,n}/c) = W_1\sqrt{6}/(27.5/2.57) \doteq 5.27$$

is highly significant it is desirable to proceed further and test the next largest difference. Hence

$$q = W_2\sqrt{6}/(27.5/2.57) \doteq 3.20$$

is the appropriate test to make. Since this value is nonsignificant the conclusion is reached that while $\mu_2 \neq \mu_4$ there are no detected differences among the other means.

One might also consider orthogonal polynomials if alternative approaches are desired.

POWER OF THE RANGE TEST IN ELEMENTARY DESIGNS

In this section the power of the range test in elementary designs will be compared with the power involved in the corresponding analysis of variance. From the previous examples it would seem that only a small loss in power is experienced.

Random Model

Consider the random model

$$X_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad (i = 1,\ldots,m;\ j = 1,\ldots,n) \tag{24}$$

where μ is a constant and the α_i, ϵ_{ij} are all mutually independent normal random variables with variances σ^2 (for α) and σ_0^2 for ϵ. Thus σ^2 is the variance between the groups and σ_0^2 is the variance within the groups, then the observed group means, $\bar{X}_{i.}$, will have a variance of $\sigma^2 + \sigma_0^2/n$ instead of σ_0^2/n. Hence the usual ratio of the between-groups mean square to the within-groups mean square is distributed not as F; but, as $(1 + n\theta^2)F$, where $\theta = \sigma/\sigma_0$ the degrees of freedom of F being $v_1 = m - 1$ and $v_2 = m(n - 1)$.

When range methods are used the null hypothesis is tested by referring the ratio

$$q = \sqrt{n}\ \text{range}\ (\bar{X}_{i.})/(\bar{w}/c)$$

to tables of the 'Studentized' range. By following a similar argument it will be seen that the presence of the α-terms multiplies the standard deviation of $\sqrt{n\bar{X}_{i.}}$ by $\sqrt{1 + n\theta^2}$; thus q_α will be distributed approximately as $\sqrt{1 + n\theta^2}q$, where q has sample size n and degrees of freedom v.

Define β to be the probability of a type II error occurring and define $1 - \beta$ to be the power involved for a particular test. To compare the power of the F-test with the power of the q-test assign a significance level α and find the value of θ for which the power of the F-test has a selected value $1 - \beta$; then find the power of the q-test for the same value of θ. If $F(\alpha;v_1,v_2)$ is the upper $100\alpha\%$ point of F for degrees of freedom v_1, v_2 then

$$1 + n\theta^2 = F(\alpha;v_1,v_2)/F(1 - \beta;v_1,v_2)\ .$$

The power of the q-test for the same value of θ is

$$P\left[\sqrt{1+n\theta^2}\,q \geq q(\alpha;m,v)\right] = P\left[q \geq q(\alpha;m,v)\sqrt{F(1-\beta;v_1,v_2)/F(\alpha;v_1,v_2)}\right]$$

The results are given below from (David, 1953, p. 348).

Table 20

Power of the q-test for a C.R.D. (Random Model)
when Power of corresponding F-test = 1 - B

20a. $\alpha = 0.05$, $1 - \beta = 0.90$

m \ n	3	6	∞
4	0.90	0.90	0.90
6	0.86	0.88	0.89
8	0.85	0.87	0.87
10	0.83	0.86	0.86

20b. $\alpha = 0.01$, $1 - \beta = 0.90$

m \ n	3	6	∞
4	0.89	0.89	0.89
6	0.87	0.88	0.88
8	0.85	0.86	0.87
10	0.83	0.85	0.85

20c. $\alpha = 0.05$, $1 - \beta = 0.75$

m \ n	3	6	∞
4	0.74	0.74	0.74
6	0.68	0.72	0.73
8	0.67	0.69	0.69
10	0.65	0.69	0.69

20d. $\alpha = 0.01$, $1 - \beta = 0.75$

m \ n	3	6	∞
4	0.73	0.74	0.74
6	0.70	0.71	0.72
8	0.68	0.69	0.69
10	0.64	0.67	0.67

From the table it is obvious that the loss of power is small, especially in common situations where the number of groups m is not very large and the number of observations per group n is not very small. It can be seen that the number of groups has more effect on the power than does the number of observations per group.

Very similar results to those obtained in Table (20) are obtained for the randomized complete block design;

$$X_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \qquad (i = 1,\ldots,m;\ j = 1,\ldots,n) \tag{25}$$

In the following table n has been taken as 4, 7, and ∞ so that the F-ratios will have degrees of freedom corresponding to those of Table (20).

Table 21

Power of the q-test for a Double Classification into m Treatments and n Blocks (Random Model)

21a. $\alpha = 0.05$, $1 - \beta = 0.90$

m	n = 4	n = 7	n = ∞
4	0.90	0.90	0.90
6	0.88	0.89	0.89
8	0.87	0.87	0.87

21b. $\alpha = 0.01$, $1 - \beta = 0.90$

m	n = 4	n = 7	n = ∞
4	0.90	0.89	0.89
6	0.89	0.88	0.88
8	0.87	0.87	0.87

21c. $\alpha = 0.05$, $1 - \beta = 0.75$

m	n = 4	n = 7	n = ∞
4	0.75	0.74	0.74
6	0.71	0.72	0.73
8	0.71	0.71	0.71

21d. $\alpha = 0.01$, $1 - \beta = 0.75$

m	n = 4	n = 7	n = ∞
4	0.75	0.74	0.74
6	0.73	0.71	0.72
8	0.71	0.70	0.69

Fixed Model

Consider the model

$$X_{ij} = \mu + A_i + \epsilon_{ij} \quad (i = 1,\ldots,m;\ j = 1,\ldots,n). \tag{26}$$

If the fixed model is assumed then the alternative hypothesis would state that the groups have means with different expectations but the same variance. Thus their range may be termed a non-central range and so q would be distributed as the ratio of a non-central range to the average of a certain number of central ranges. The distribution function of a non-central range is the sum of a number of multiple integrals and is extremely complicated and thus the distribution of the ratio of non-central range to mean-central range is very difficult and has not been worked out to my knowledge.

SUMMARY

Overall little loss in statistical efficiency results from the use of mean range in place of error mean square s^2. Under certain conditions the

range of treatment means is more powerful in detecting certain patterns of treatment differences than is the treatment mean square while for other patterns the situation is reversed.

The reduction in computational labor is often considerable but the advantage of this is sometimes offset by the fact that the analysis of variance technique is often handled by computors while the analysis of range often requires an initial period of learning.

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APPENDIX

Table I.

Scale Factor c and Equivalent Degrees of Freedom v Appropriate to the Mean of m Uncorrelated Ranges of n Observations

	n = 2		3		4		5		6	
m	v	c	v	c	v	c	v	c	v	c
2	1.9	1.28	3.8	1.81	5.7	2.15	7.5	2.40	9.2	2.60
3	2.8	1.23	5.7	1.77	8.4	2.12	11.1	2.38	13.6	2.58
4	3.7	1.21	7.5	1.75	11.2	2.11	14.7	2.37	18.1	2.57
5	4.6	1.19	9.3	1.74	13.9	2.10	18.4	2.36	22.6	2.56
10	9.0	1.16	18.4	1.72	27.6	2.08	36.5	2.34	44.9	2.55
	.88	1.13	1.82	1.69	2.74	2.06	3.62	2.33	4.47	2.53

Table II.

Scale Factor c and Equivalent Degrees of Freedom v for Analysis of Double Classification

	n = 2		3		4		5		6	
k	v	c	v	c	v	c	v	c	v	c
2	1.0	1.00	2.0	1.35	2.9	1.58	3.8	1.75	4.7	1.89
3	1.9	1.05	3.7	1.48	5.6	1.76	7.4	1.96	9.3	2.12
4	2.7	1.07	5.4	1.54	8.2	1.84	11.0	2.06	13.9	2.23
5	3.6	1.08	7.2	1.57	10.9	1.88	14.6	2.12	18.5	2.30
6	4.5	1.09	8.9	1.59	13.6	1.91	18.2	2.15	23.0	2.34
7	5.4	1.09	10.7	1.61	16.3	1.93	21.8	2.18	27.6	2.37
8	6.3	1.10	12.5	1.62	19.0	1.95	25.4	2.20	32.1	2.39
9	7.1	1.10	14.3	1.63	21.7	1.96	29.0	2.21	36.6	2.41
10	8.0	1.10	16.1	1.63	24.4	1.97	32.6	2.22	41.9	2.42
20	8.9	1.11	33.9	1.66	51.5	2.02	68.8	2.28	86.0	2.48
	.87	1.13	1.80	1.69	2.71	1.06	3.62	2.33	4.50	2.53

Note: The last line in each of the above tables is the constant difference at infinity.

Table III.

Scale Factor c and Equivalent Degrees of Freedom v = lv' for the Split-plot Design with l Main Treatments, m Blocks and n Subtreatments

	n							
	3		4		5		6	
m	v'	c	v'	c	v'	c	v'	c
2	1.9	1.23	2.8	1.48	3.7	1.67	4.6	1.81
3	3.6	1.40	5.4	1.70	7.2	1.92	9.2	2.08
4	5.3	1.48	8.1	1.80	10.9	2.04	13.8	2.22
5	7.0	1.53	10.7	1.86	14.5	2.10	18.3	2.28
6	8.8	1.56	13.4	1.90	18.1	2.14	22.9	2.33
7	10.6	1.59	16.2	1.92	21.7	2.17	27.5	2.36
8	12.3	1.60	18.8	1.94	25.2	2.19	32.1	2.38

Table IV.

Weighting Factors d_n, d_n/V_n and d_n^2 for Analysis of Single Classification with Unequal Cell Frequencies

	n							
	3	4	5	6	7	8	9	10
d_n	1.69	2.06	2.33	2.53	2.70	2.85	2.97	3.08
$\frac{d_n}{V_n}$	2.15	2.66	3.12	3.52	3.90	4.24	4.55	4.84
$\frac{d_n^2}{V_n}$	3.63	5.48	7.25	8.93	10.54	12.06	13.51	14.90

THE APPLICATION OF RANGE IN THE

OVERALL ANALYSIS OF VARIANCE

by

DARREL LEE EKLUND

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One of the best known estimators of the variation within a sample is the sample range. Although it has been widely used in industrial quality control, its application to the analysis of experimental data has been limited in favor of the statistically more efficient, but computationally more tedious sample variance. Part of the reason for this is that the analysis of variance was first developed in connection with experiments in which the computational labor of analysis often represented only a small fraction of the labor of experimentation. Hence the maximum amount of information was used in the analysis of the data. Now the analysis of variance is more widely used, but situations frequently arise in which data are cheap and time available for analysis often is limited.

The purposes for which one may wish to use such a short-cut measure are two-fold:

1) In large scale analysis of data one may wish to save computational labor by basing the analysis completely on the short-cut measure.

2) It serves as a quick and independent computational check on a full mean square analysis of variance.

W. S. Gosset who wrote under the pseudonym of 'Student' is given credit for proposing the use of the ratio of the range divided by an independent estimate s of the population standard deviation. As early as 1932 he referred to this ratio as the 'Studentized' ratio in a letter written to E. S. Pearson. In 1944, J. W. Rodgers showed how to utilize the range in estimating all of the variances involved in the analysis of variance and he credited W. J. Jennett for suggesting the procedure. P. B. Patnaik (1950) developed the theory and procedure for the utilization of the range in analysing a completely randomized design. H. O. Hartley (1950) modified the procedure to show how a randomized complete block design might be handled. In 1951,

H. A. David concluded the range analysis procedures by presenting procedures for the completely randomized design with cell replication and factorial arrangement of treatments, the randomized complete block design with factorial arrangement of treatments, the split-plot design and gave an approximate method of dealing with a completely randomized design with unequal cell frequencies.

The purpose of this report is to review the range analysis methods used in the above designs, present them in collected form with an example illustrating the procedure involved in each case and to get an idea of the power which may be attributed to these tests.